

POTWI 14 SOLUTION

Key fact. The triangle whose sides are the three medians of a triangle T has area $\frac{3}{4}$ of the area of T .

Proof of the fact. Put the vertices at position vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$. The medians (as vectors from each vertex to the midpoint of the opposite side) are

$$\mathbf{m}_a = \frac{1}{2}(\mathbf{b} + \mathbf{c}) - \mathbf{a}, \quad \mathbf{m}_b = \frac{1}{2}(\mathbf{a} + \mathbf{c}) - \mathbf{b}, \quad \mathbf{m}_c = \frac{1}{2}(\mathbf{a} + \mathbf{b}) - \mathbf{c},$$

and $\mathbf{m}_a + \mathbf{m}_b + \mathbf{m}_c = \mathbf{0}$, so they do form a closed triangle. Take $\mathbf{a} = \mathbf{0}$: then $\mathbf{m}_a = \frac{1}{2}(\mathbf{b} + \mathbf{c})$ and $\mathbf{m}_b = \frac{1}{2}(\mathbf{c} - 2\mathbf{b})$, and

$$\mathbf{m}_a \times \mathbf{m}_b = \frac{1}{4}(\mathbf{b} + \mathbf{c}) \times (\mathbf{c} - 2\mathbf{b}) = \frac{1}{4} \cdot 3(\mathbf{b} \times \mathbf{c}).$$

So the median-triangle has area $\frac{1}{2} \cdot \frac{3}{4} |\mathbf{b} \times \mathbf{c}| = \frac{3}{4} [T]$.

Apply it. The triangle with sides 9, 12, 15 is right-angled, since $9^2 + 12^2 = 81 + 144 = 225 = 15^2$. Its area is $\frac{1}{2} \cdot 9 \cdot 12 = 54$. Hence

$$[T] = \frac{4}{3} \cdot 54 = 72.$$

Answer: (D) 72
