

POTWI 12 SOLUTION

Label the vertices v_1, v_2, v_3, v_4, v_5 in order around the pentagon, so v_5 is also adjacent to v_1 .

Step 1: colour the vertices as a path. Ignore the edge v_5v_1 for now. Choose v_1 in 3 ways, and each later vertex must differ from the one before it, giving 2 choices each:

$$3 \cdot 2^4 = 48.$$

Among these 48, some have $v_5 = v_1$, which is not allowed. We count and remove them.

Step 2: colourings with $v_5 = v_1$. If v_5 has the same colour as v_1 , we may merge them into one vertex. The condition that $v_4 \neq v_5$ becomes $v_4 \neq v_1$, so these colourings correspond exactly to proper colourings of the 4-cycle $v_1v_2v_3v_4$.

Step 3: count the 4-cycle. Colour $v_1, \dots, v_4$ as a path: $3 \cdot 2^3 = 24$. Those with $v_4 = v_1$ correspond (by merging again) to proper colourings of a triangle, of which there are $3! = 6$. So the 4-cycle has $24 - 6 = 18$ proper colourings.

Step 4: finish.

$$48 - 18 = 30.$$

(This agrees with the chromatic polynomial of a cycle, $(k-1)^n + (-1)^n(k-1)$, which for $n = 5, k = 3$ gives $3^2 - 2 = 30$.)

Answer: (A) 30