24, 12, 6, ...

Problem of the Week
Problem E and Solution
Summing up a Sequence 2

Problem

The first term in a sequence is 24. We can determine the next terms in the sequence as follows:

- If a term is even, then divide it by 2 to get the next term.
- If a term is odd, then multiply it by 3 and add 1 to get the next term.

By doing this, we can determine that the first three terms in the sequence are 24, 12, and 6.

Shweta writes the first n terms in this sequence and notices that the sum of these terms is a four-digit number. How many different possible values of n are there?

Solution

We will begin by finding more terms in the sequence. The first 14 terms of the sequence are 24, 12, 6, 3, 10, 5, 16, 8, 4, 2, 1, 4, 2, 1.

If we continue the sequence, we will see that the terms 4, 2, and 1 will continue to repeat. Now we want to find the smallest and largest possible values of n so that the sum of the terms in the sequence from term 1 to term n is a four-digit number. We will start by finding the smallest possible value of n.

The sum of the first 8 terms is $24 + 12 + 6 + 3 + 10 + 5 + 16 + 8 = 84$. The sum of the repeating numbers is $4 + 2 + 1 = 7$. We want to determine the number of groups of repeating numbers. Let this be g. Suppose $84 + 7g = 1000$. Solving this gives $7g = 916$, so $g \approx 130.857$.

If $g = 130$, then the sum of the terms in the sequence is $84 + 7 \times 130 = 994$. This sequence contains the first 8 terms, plus 130 groups of the three repeating numbers. Therefore there are a total of $8 + 3 \times 130 = 398$ terms.

The 399^{th} term in the sequence will be 4, so the sum of the first 399 terms will be $994 + 4 = 998$.

The 400^{th} term in the sequence will be 2, so the sum of the first 400 terms will be $998 + 2 = 1000$. This is the smallest possible four-digit number, so the smallest possible value of n is 400.

Now we will find the largest possible value of n. Using a similar approach, let g be the number of groups of repeating numbers. Suppose $84 + 7g = 9999$. Solving this gives $g \approx 1416.429$.

If $g = 1416$, then the sum of the terms in the sequence is $84 + 7 \times 1416 = 9996$. This sequence contains the first 8 terms, plus 1416 groups of the three repeating numbers. Therefore there are a total of $8 + 3 \times 1416 = 4256$ terms.

The 4257^{th} term in the sequence will be 4, so the sum of the first 4257 terms will be $9996 + 4 = 10\,000$. Since this is not a four-digit number, the largest possible value of n is 4256.

So n can be any positive integer between 400 and 4256, inclusive. This is a total of $4256 - 400 + 1 = 3857$ possible values.

Extension:

In 1937, the mathematician Lothat Collatz wondered if any sequence whose terms after the first are determined in this way would always eventually reach the number 1, regardless of which number you started with. This problem is actually still unsolved today and is called the Collatz Conjecture.