Problem of the Week

$t_n = ar^{n-1}$

Problem E and Solution

A Geometric Problem

Problem

The first term in a geometric sequence is a, the second term is b, and the third term is c. The three terms have a sum of 158 and a product of 74 088.

Determine all possible ordered triples (a, b, c).

Solution

Let r be the common ratio of the geometric sequence. Since a is the first term of the sequence, then $b = ar$ and $c = ar^2$.

We are given that $abc = 74\,088$. Thus, $a(ar)(ar^2) = a^3r^3 = (ar)^3 = 74\,088$. Therefore, $ar = 42$. Since $b = ar$, we have $b = 42$.

Now, $a + b + c = 158$ becomes $a + 42 + c = 158$, or $a + c = 116$.

Since $b = ar$, then $42 = ar$, or $r = \dfrac{42}{a}$ (since the product of a, b, and c is not zero, we know $a \neq 0$).

Therefore, $c = ar^2 = a\left(\dfrac{42}{a}\right)^2 = a\left(\dfrac{1764}{a^2}\right) = \dfrac{1764}{a}$.

Substituting $c = \dfrac{1764}{a}$ into $a + c = 116$, we have

$$\begin{aligned} a + \frac{1764}{a} &= 116 \\ a^2 + 1764 &= 116a \\ a^2 - 116a + 1764 &= 0 \\ (a - 18)(a - 98) &= 0 \end{aligned}$$

Therefore, $a = 18$ or $a = 98$.

When $a = 18$, then $r = \frac{42}{18} = \frac{7}{3}$, and one ordered triple is (18, 42, 98).
Indeed, we can check that $18 + 42 + 98 = 158$ and $(18)(42)(98) = 74\,088$.

When $a = 98$, then $r = \frac{42}{98} = \frac{3}{7}$, and one ordered triple is (98, 42, 18).
Indeed, we can check that $98 + 42 + 18 = 158$ and $(98)(42)(18) = 74\,088$.

In conclusion there are two ordered triples that satisfy the conditions of the problem. They are (18, 42, 98) and (98, 42, 18).